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ORIGINAL ARTICLE

Plural Logicism

Francesca Boccuni

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Abstract PG (*Plural Grundgesetze*) is a consistent second-order system which is aimed to derive second-order Peano arithmetic. It employs the notion of plural quantification and a few Fregean devices, among which the infamous Basic Law V. George Boolos' plural semantics is replaced with Enrico Martino's *Acts of Choice Semantics* (ACS), which is developed from the notion of arbitrary reference in mathematical reasoning. Also, substitutional quantification is exploited to interpret quantification into predicate position. ACS provides a form of logicism which is radically alternative to Frege's and which is grounded on the existence of individuals rather than on the existence of concepts.

It is well-known that Frege's logicist foundation of mathematics exposed in *Grundgesetze der Arithmetik* is inconsistent. The contradiction is derived from the infamous Basic Law V. This principle is crucial to Frege's logicism as it embeds the tenet that tightly connects natural numbers, conceived as equivalence classes, to concepts. Since, according to Frege, extensions are logically dependent on concepts, numbers as extensions inherit their logicality from that relation of logical dependence. The failure of his programme doomed the possibility of deriving arithmetic on purely logical basis, where the overall logicality of the programme was embedded in the logical connection between concepts and extensions.

In this article, I shall present a predicative second-order system, *Plural Grundgesetze* (PG), which interprets second-order Peano arithmetic. The main features of PG are plural quantification, which guarantees the strength of full second-order logic to PG, and

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a particular semantics, the *Acts of Choice Semantics* (ACS).[1] I will show that, on the grounds of ACS, PG embodies a form of logicism which is radically different from Frege's, as it is grounded on the existence of individuals rather than on the existence of concepts.

1 Plural Grundgesetze: A System

The basic features of the language $\mathscr{L}$ of PG are:

(i) an infinite list of singular individual variables $x, y, z, \ldots$;
(ii) an infinite list of plural individual variables $X, Y, Z, \ldots$, that vary *plurally* over the individuals of the first-order domain;
(iii) an infinite list of monadic predicate variables $F, G, H, \ldots$;
(iv) the logical constants $\neg, \rightarrow, =$;
(v) existential quantifiers $\exists$ for every kind of variables;
(vi) the constant relation symbol η;
(vii) the abstraction operator $\{:\}$.

The atomic formulæ of $\mathscr{L}$ are:

(viii) $\mathbf{a} = \mathbf{b}$;
(ix) $\mathbf{a}\eta\mathbf{Y}$;[2]
(x) $\mathbf{Pa}$,

where **a** and **b** are metavariables for the terms of $\mathscr{L}$, **Y** is a metavariable for plural variables and **P** is a metavariable for predicate variables. Formulæ of kind (ix) express what I may call *plural reference*, meanwhile formulæ of kind (x) express regular predication. Primitive existential quantification for every kind of variables is available. Universal quantification for every kind of variables can be defined in the obvious way.

Along with the singular variables $x, y, z, \ldots$, the first-order *terms* of $\mathscr{L}$ are:

(xi) an infinite list of extension-terms of the form $\{x{:}\psi x\}$,

where ψ is a formula of $\mathscr{L}$ containing neither bound predicate variables nor free plural variables. It may contain, though, free predicate variables, bound plural variables, and both free and bound singular variables. Also, nested extension-terms may appear in extension-terms.

Two Comprehension Principles are available in PG: a *Plural Comprehension Principle*

$$\text{(PLC)} \quad \exists X \forall x (x \eta X \leftrightarrow \phi x),$$

[1] See also Boccuni (2010) for PG with a mixed Boolos-Fregean semantics. There are two reasons of discontent with that theory: first, it may be quite disputed whether it embodies some form of logicism; secondly, the Julius Caesar problem arises. We shall see in what follows that the present theory with ACS solves both issues. See also Boccuni et al. (2012) on this.

[2] To be read "**a** is among the **Y**s".

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$$\text{(PRC)} \quad \exists F \forall x(Fx \leftrightarrow \psi x),$$

where ψ contains neither F free, nor free plural variables, nor bound predicate variables. A schematic formulation of *Basic Law V* is also among the axioms:where ϕ does not contain X free; and a *Predicative Comprehension Principle*

(V)

$$\{x : \psi x\} = \{x : \chi x\} \leftrightarrow \forall x(\psi x \leftrightarrow \chi x)^{3} \qquad \text{(V)}$$

Axiom V guarantees the existence of Dedekind-infinitely many first-order individuals in the domain. This is crucial to guarantee that Peano axioms may be derived in PG. I'll say more about the restrictions on PRC later.

It is worth noticing that the restrictions on the formulæ permitted on the right-hand side of PRC are exactly the same restrictions imposed on the formulæ permitted in the extension-terms. This guarantees that in PG there is a one-to-one correspondence between predicates and extension-terms.[4]

2 Peano Axioms

A few more definitions are needed in order to derive Peano axioms. The *singleton* and the notion of *unordered pair* may be defined as usual:

Definition 1 $\{x\} =_{def} \{y: x = y\}$;

Definition 2 $\{x, y\} =_{def} \{z: z = x \vee z = y\}$.

The usual Wiener-Kuratowski definition of the *ordered pair* easily follows:

Definition 3 $(x, y) =_{def} \{\{x\}, \{x, y\}\}$.[5]

Notice that, strictly speaking, $\mathscr{L}$ is monadic. The introduction of pairs, nevertheless, provides $\mathscr{L}$ with *polyadic* expressive capacity: the formula $F(x, y)$, in fact, means that the individual (x, y) satisfies the predicate F, and the formula $(x, y)\eta Y$ means that the individual (x, y) is among the Ys.

In $\mathscr{L}$, *natural numbers* may be defined inductively. The individual constant "0" may be introduced by definition:

[3] For some similarities with PG, see Burgess (2005, 2.3d), where a second-order language with a full second-order comprehension axiom for concepts in general, a predicative second-order comprehension axiom, and an axiom stating that, to every predicative concept, there corresponds an extension is sketched. In this setting, not all definable concepts interact with extensions—some of them "float" over extensions. Analogously, in PG not all pluralities interact with extension-term formation.

[4] For a proof of model-theoretic consistency for PG, see Boccuni (2011a). The consistency of PG is indeed a remarkable result. In fact, it has been argued that second-order system with Basic Law V beyond Δ^1_1-comprehension are inconsistent. The consistency of PG is remarkable in that it makes Σ^1_1- and Π^1_1- plural formulæ safely interact with Axiom V. See Boccuni (2011a) also for some considerations on the mathematical strength of PG, which is likely equi-consistent with PA^2.

[5] The fundamental law of the ordered pair $(x, y) = (u, v) \leftrightarrow x = u \wedge y = v$ may be easily derived in PG, through several applications of the usual rules of inference, axiom V, and the definitions of the unordered and ordered pairs.

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Definition 4 $0 =_{def} \{x\colon x \neq x\}$.

Consequently, numbers may be inductively defined:

Definition 5 $1 =_{def} \{x\colon x = 0\}$;

Definition 6 $2 =_{def} \{x\colon x = 1\}$;

and so on. In general, the *successor* of a number is its singleton. In this way, we get the usual Zermelo natural numbers.

A plurality X is *inductive* whenever it contains 0 and it is closed under the successor. The usual definition of the set of natural numbers may be given in terms of pluralities. First, a predicate $\mathbb{N}$ is defined:

Definition 7 $\mathbb{N}x \leftrightarrow_{def} \forall Y(Y \text{ is inductive } \rightarrow x\eta Y)$.

Given the previous definitions, the following formulations of second-order Peano axioms are derivable in PG, with the singular variables x and y restricted to $\mathbb{N}$:

Theorem 1 $\mathbb{N}0$

Proof That 0 is a number trivially follows from the definition of $\mathbb{N}$.

Theorem 2 $\forall x(\{x\} \neq 0)$

Proof Let us assume that there is an individual y such that $\{y\} = 0$. On the grounds of the definition of 0, thus, y must satisfy the condition $x \neq x$. As no individual is not self-identical, 0 is no successor.[6]

Theorem 3 $\forall xy(\{x\} = \{y\} \rightarrow x = y)$

Proof Let x and y be two arbitrary individuals of the first-order domain of $\mathscr{L}$. If $\{x\} = \{y\}$, then, on the grounds both of axiom V and of the definition of the singleton, for all $z, z = x \leftrightarrow z = y$. Thus, for the transitivity of identity, $x = y$. As x, y are arbitrary, the generalization $\forall xy(\{x\} = \{y\} \rightarrow x = y)$ is valid.

Theorem 4 $\forall X(0\eta X \wedge \forall x(x\eta X \rightarrow \{x\}\eta X) \rightarrow \forall x(x\eta X))$

Proof It trivially follows from the definition of $\mathbb{N}$.

It has to be stressed that the derivation of PA^2 from PG is rather unFregean, as, in fact, it does not proceed by Hume's Principle. PG is unFregean also under another respect: the usual Fregean definition of the concept of predecessor does not seem to follow from PG's definitions, at least not in a straightforward manner. The introduction of such a concept, in fact, would require an impredicative second-order quantification, which is not available in PRC. Thus, the recovery of Frege

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[6] The formal proof of this theorem makes a crucial use also of Axiom V and of the definition of the singleton.

Proof 1(1)$\exists y(\{y\} = 0)\mathscr{A}$ 2(2)$\{a\} = 0$ $\mathscr{A}$2(3)$\{x : x = a\} = \{x : x \neq x\}$ 2, Def. $\{a\}$and 0(4)$\{x : x = a\} = \{x : x \neq x\} \leftrightarrow \forall x(x = a \leftrightarrow x \neq x)$ Axiom V (5)$(\{x : x = a\} = \{x : x \neq x\} \rightarrow \forall x(x = a \leftrightarrow x / = x)) \wedge (\forall x(x = a \leftrightarrow x \neq x) \rightarrow \{x : x = a\} = \{x : x \neq x\})$ 4 Def $\leftrightarrow$ (6)$\{x : x = a\} = \{x : x \neq x\} \rightarrow \forall x(x = a \leftrightarrow x \neq x)$ 5, E $\wedge$ 2(7) $\forall x(x = a \leftrightarrow x \neq x)$ 3, 6 MP 2(8)$(a = a) \leftrightarrow (a \neq a)$ 7, EU 1(9)$(a = a) \leftrightarrow (a \neq a)$ 2, 8 EE (10)$\neg\exists y(\{y\} = 0)$ 1, 9 RAA (11)$\forall y \neg(\{y\} = 0)$ 10 by $\neg\exists x\phi x \equiv \forall x \neg \phi x$(12) $\forall y(\{y\} \neq 0)$ 11 by the usual definition of ‘ $\neq$

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Arithmetic does not seem easily viable in PG, though Frege Arithmetic is interpretable in PG since it is interpretable in PA^2. This, of course, is a matter of the definitions actually in PG. Some more complicated ones could be provided for explicitly recovering Frege Arithmetic. These peculiarities of PG are not at odds with my main aim, since PG is meant to embody a form of logicism that, unlike Frege's, is not grounded on a theory of concepts, but rather on a theory of individuals accounted for by ACS.

3 The Theory of Ideal Reference

According to Martino (2001, 2004), the possibility of directly referring, at least ideally, to any object of a universe of discourse is presupposed both by logical and mathematical reasoning, even when non-denumerable domains are concerned Martino (2001, 2004) call this claim the *Thesis of Ideal Reference* (TIR).[7] Such a possibility of direct reference is very well expressed by the crucial role *arbitrary reference* plays both in formal and informal reasoning. Its cruciality lies in that arbitrary reference exhibits two different logical features that make it essential for performing proofs, i.e. *arbitrariness* and *determinacy*. Through arbitrary reference, we may consider *any* object *a* of a universe of discourse. Consequently, the arguments about *a* retain their general validity. At the same time, though, within the arguments about it, "*a*" is required to denote a determinate object, distinct from all the other objects in the domain it belongs to.

In order to motivate TIR, an account of the genuine referentiality of arbitrary reference and its directness has to be provided. It may be argued, in fact, that arbitrary reference is not genuine, since free variables and arbitrary names do not refer at all.[8] Evidence in favour of the genuine referentiality of arbitrary reference may be found in Boccuni (2010), Breckenridge and Magidor (2012), and Martino (2001, 2004). In this Section, I will try to provide a more general argument to this aim. My claim is that the validity of arguments in mathematical and logical reasoning requires the underlying assumption of the genuine referentiality of arbitrary reference. The relation between validity and referentiality will be accounted for in terms of sameness and determinacy of reference.

Usually, an arbitrary name "*a*" is used to refer to the *same* individual *a* within a derivation on *a*. A crucial reason for this is to be found in the validity of (some) argument schemas. If sameness of reference were not a basic ingredient of derivations, validity would be in jeopardy.[9] Consider the rule of existential elimination in natural deduction. When we pass from a premise of the form $\exists x \, \phi \, x$ to the auxiliary assumption $\phi \, (a)$, "*a*" has to be an unused arbitrary name, or at least it has not to appear in any of the assumptions which $\exists x \, \phi \, x$ depends upon. Consider now the following (invalid) deduction:

[7] See also Boccuni (2010) and Carrara and Martino (2010) for further applications of TIR.

[8] See, for instance, Pettigrew (2008).

[9] A further argument to this aim, from the uniformity of substitution of predicate and individual letters in argument schemas, may be found in Boccuni (2010).

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(1) $\exists x Hx \quad \mathscr{A}$
(2) $\exists x \neg Hx \quad \mathscr{A}$
(3) $Ha \quad \mathscr{A}$
(4) $\neg Ha \quad \mathscr{A}$
(5) $Ha \wedge \neg Ha \quad 3, 4$ intr. $\wedge$

Invalidity stems out from that, in eliminating the existential quantifiers respectively from (1) and (2), we use the very same arbitrary name in (3) and (4).[10] Say that H is the property of being even and x varies over the natural numbers: (1) and (2) respectively say that there is at least a number which is even and there is at least a number which is not. Both these sentences are true in the standard model of Peano arithmetic. Nevertheless, if we use the same arbitrary name to perform existential elimination in the derivation above, in (3) and (4) we respectively say that a number is even and that *the very same number* is not, from which the contradiction in (5). For this reason, using an already used arbitrary name in (4) cannot be allowed.

In order to explain the invalidity of the derivation (1)–(5) "*a*" must be referring to the same, though arbitrary, individual both in lines (3) and (4). Thus, in order to achieve validity in the previous example, in line (4) we have to use a different arbitrary name than "*a*", because we need to express that a *different* individual than a is $\neg H$ within the same derivation, according to the restrictions imposed on universal introduction and existential elimination. But then again, in order to distinguish between a and any other arbitrary individual that is $\neg H$, we have to assume that a is a *determinate*, though arbitrary, individual of the domain. The motivation for this requirement is very nicely explained by Suppes:

> (…) ambiguous names,[11] like all names, cannot be used indiscriminately. The person who calls a loved one by the name of a *former* loved one is quickly made aware of this. (…) Such a happy-go-lucky naming process is bound to lead to error, just as we could infer a false conclusion from true facts about two individuals named "Fred Smith" if we did not somehow devise a notational device for distinguishing which Fred Smith was being referred to in any given statement. The restriction which we impose to stop such invalid arguments is to require that when we introduce by existential specification an ambiguous name in a derivation, that name has not previously been used in the derivation.[12]

[10] See Suppes (1999, p. 82) for this example.

[11] I.e. arbitrary names.

[12] Suppes (1999, p. 82). Of course, it is not always the case that using the same arbitrary name leads to invalidity, nor that different arbitrary names have to refer to different individuals. For instance, consider using "*a*" for eliminating the quantifiers both from $\forall x Fx$ and $\forall x Gx$ in the same derivation, where x varies over the natural numbers and both formulæ have a model in Peano arithmetic. Or consider using "*a*" and "*b*" for eliminating respectively the first quantifier and the second, where a and b may well be the same individual. In none of these examples, sameness of reference seems to lead to invalidity, but such an innocuousness does not by itself speak against the genuine referentiality of "*a*" or the importance of sameness of reference to derivations. It rather testifies that there are contexts in which the co-referentiality of all the occurrences of "*a*" (or of "*a*" and "*b*", for that matter) is not problematic.

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The reasons for restricting the rules of introduction and elimination of quantifiers in natural deduction are semantic, namely we perform some underlying semantic reasoning in derivations which we want to be captured by deductive rules and restrictions on them. Such a reasoning is crucially based on sameness and determinacy of reference of arbitrary names. But then again, in order to make sense of sameness and determinacy, and consequently of the requirements we pose on deductive rules for the sake of validity, we have to assume the genuine referentiality of arbitrary names. Genuine referentiality, then, is a necessary condition for validity. This relation can be highlighted by investigating the role that sameness and determinacy of arbitrary reference have in derivations. In fact, if "*a*" were not referential at all, how could we account for *a* being the same individual throughout an argument? Those who support the non-referentiality of arbitrary reference should provide some argument for explaining how, then, formal and informal reasoning functions in the way it does (for instance, by certain constraints on introduction and elimination of quantifiers).

In spite of the previous considerations, genuine referentiality may still sound at odds with arbitrariness. I will show that this is not the case by clarifying what arbitrariness in arbitrary reference amounts to. One possible way to see this issue is to claiming that what is arbitrary is the reference relation itself. For instance, Russell (1967), on discussing the role of the free variables in mathematical reasoning, writes

> If we say: "Let *ABC* be a triangle, then the sides *AB* and *AC* are together greater than the side *BC*", we are saying something about *one* triangle, not about *all* triangles; but the one triangle concerned is absolutely ambiguous, and our statement consequently is also absolutely ambiguous.[13]

When Russell speaks of ambiguous names, he seems to have in mind that reference is ambiguous. And indeed

> Naturally we have no definite individual in mind when we use "John Doe", and it may properly be claimed that "John Doe" is not a genuine proper name; that is why we use the terminology "ambiguous name".[14] Nevertheless, "*a*" has to refer to a determinate individual within an argument on *a*, so the reference relation between "*a*" and *a*, once established, is not ambiguous at all.

A rather different argument is by Kit Fine. According to him, arbitrariness is a property of some special kind of objects, namely those referred to by arbitrary names. To this extent, we may claim that, though *a* is an object having the property of being arbitrary, we may still determinately refer to it. Nevertheless, it is because of a property that makes *a* what it is, that we cannot say which object *a* is. Thus, *a* is *intrinsically* indeterminate, namely it is indeterminate by its own nature. This would clearly violate the requirement of *a* being a *determinate* object, which is indeed so crucial. But then again, if *a* is not determinate, then how can we be sure that "*a*" refers to the very same object throughout a derivation on *a*?

13FL01 [13] Russell (1967, pp. 156–157). "ABC" is a free variable.

14FL01 [14] Suppes (1999, p. 81).

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The best way to view arbitrariness, then, is as an *epistemic* feature: a is determinate, and "a" determinately refers to it, but we do not *know which* individual a is.[15] This interpretation, on the one hand, saves the intuition concerning generality. In a sense, our lack of knowledge of which individual a is justifies the applicability of the rule of introduction of the universal quantifier (under the usual restrictions): since a is not an individual I could pick among all others because I do not know which one it is, the conclusion I draw on a is valid for all individuals of the domain (provided that the restrictions on the rule are respected). The epistemic interpretation of arbitrariness also preserves genuineness, since I may not know which individual a is, but this is not incompatible, unlike metaphysic and semantic arbitrariness, with a being a determinate individual and thus reference to a being genuine.

The second feature of arbitrary reference that I want to stress out pertains to its relation with quantification. Consider once again the rule of existential elimination. As Martino (2004) points out, the possibility of passing from a purely existential assumption such as $\exists x \phi\, x$ to the consideration of an arbitrary object a such that $\phi\, a$ is guaranteed by the rule of elimination of the existential quantifier which allows to substitute the given existential assumption with the auxiliary assumption $\phi\, a$. If the rules of inference that govern the use of the logical constants are justified by the meaning of the constants themselves, the meaning of the existential quantifier presupposes the possibility of singularly referring, at least ideally, to any individual, and consequently existential quantification logically presupposes such a possibility of reference.[16] Thus, before we simultaneously consider several entities through quantification, we are required to be able to refer to each of them, at least ideally: quantification logically presupposes the ideal possibility of referring to each and every element of a domain, before we consider those elements through generalisation.[17]

4 The Acts of Choice Semantics

From this perspective, reference to an entity exclusively in terms of a quantification on the domain it belongs to cannot be allowed, because it is required that we are able to directly refer to that entity, even if just in an ideal way, on pain of violating TIR. As a corollary of TIR, in fact, Martino provides a re-formulation of Russell's well-known *Vicious Circle Principle* (VCP*) No universe of discourse can contain an element which we can refer to only through quantification over that universe.[18] On this perspective, Frege and Russell's descriptivist theory of reference is

[15] See Breckenridge and Magidor (2012) and Martino (2001, 2004) on the epistemic interpretation of arbitrary reference.

[16] Analogously as far as the rule of introduction for universal quantification is concerned. See Martino (2004, p. 110).

[17] For further justifications and applications of arbitrary reference, see also Breckenridge and Magidor (2012).

[18] Martino (2004, p. 119), En. transl. mine. Notice that VCP* follows from TIR also when non-denumerable domains are concerned. Even though a language may lack non-denumerably many names, TIR still holds, as the ideal possibility of directly referring to each and every individual in a

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immediately ruled out, because, in their view, reference is fixed via definite descriptions. If Russell's reading of definite descriptions is assumed, these are expressed by existential formulæ which, on the grounds of TIR, logically presuppose the possibility of referring to the individuals they are supposed to individuate. The descriptivist theory of reference is circular.[19]

A theory of *direct* reference is then needed. Nevertheless, since we are here dealing with mathematical entities, we cannot appeal to a casual connection to motivate direct reference as the usual Kripkean theory of reference does. To this extent, the direct theory of reference here at stake has to be a theory of *ideal* direct reference, based on Kripke's intuitions and nevertheless idealised in such a way to account for the semantics of mathematical discourse.

This is what Carrara and Martino (2010) and Martino (2001, 2004) provide. In order to justify the possibility of direct ideal reference, Martino proposes to imagine a series of *ideal agents* that fix the reference of the meaningful expressions of a language. The ideal agents, that are pictured as having direct access to the individuals of the universe of discourse, perform an arbitrary *act of choice* through which the reference of the meaningful expressions is fixed. We can picture agents as holding scoring paddles bearing "1" on one side of the paddle, and "0" on the other. In general, as long as singular reference is concerned, for each individual *a* of the domain, there is an agent that picks *a* as the referent of "*a*" whenever she chooses "1" relative to "*a*"; the agent does not pick *a* as the referent of "*a*" whenever she chooses "0". Clearly, there have to be as many agents as individuals; but then again, since agents are mere idealisations, there is no domain of agents at all. Even more so, we may take the first-order individuals themselves to play the role of agents. The postulation of the ideal existence of the agents is just aimed to explain how acts of reference are performed in a formal language.[20]

From this idealisation, Martino formulates the following *Principle of the Act of Choice*:

(CAP) Every individual of any domain of quantification is capable of being chosen by an ideal agent.[21]

In order to obtain the appropriate semantics for second-order variables, the singular ACS has to be extended to *plural* ACS. This is also provided by Martino (2001, 2004) through the notion of act of *simultaneous* choice. By an act of simultaneous choice it is meant a simultaneous choice between the values 0, 1 performed by each agent. In this way, each agent performs a merely singular choice, meanwhile the simultaneousness guarantees that such acts involve several individuals at once. An individual is, then, designated in an act of simultaneous choice, whenever the corresponding agent chooses 1 in the relative act of choice; it is not designated otherwise.

Footnote 18 continued
non-denumerable domain may be performed via arbitrary reference, as in the case of, e.g. "let *a* be an arbitrary real number".

[19] See Martino (2001, 2004).

[20] See Martino (2004, pp. 112–113) on this.

[21] Martino (2004, p. 112), En. trans. mine.

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ACS is used in order to provide the truth-clauses for the formulæ of $\mathscr{L}$ containing singular and plural quantification, whereas an assignment function Σ provides predicate quantification with a substitutional interpretation. Let $\mathscr{D}$ be a non-empty domain of individuals. For each term t_i, consider a singular choice t_i^* of an individual of $\mathscr{D}$, for $i = 1, \ldots, n, \ldots$; for each plural variable X_j, consider a simultaneous plural choice X_j^* of individuals of $\mathscr{D}$, for $j = 1, \ldots, m, \ldots$. The truth-clauses for singular and plural quantification are inductively given, then, in terms of the acts of choice $t_1^*, \ldots, t_n^*, \ldots, X_1^*, \ldots, X_m^*, \ldots$.

Let us, then, introduce an assignment function Σ. Let us define the set β as the set of all formulæ of $\mathscr{L}$ such that they contain x as the only free variable and they do not contain bound predicate variables. Now, $\Sigma : \{F \in \mathscr{L} : F$ is a predicate variable $\} \to \beta$.

Let **a** and **b** be metavariables for the terms of $\mathscr{L}$, namely metavariables for first-order variables and extension-terms of $\mathscr{L}$; **Y** a metavariable for plural variables; **P** a metavariable for predicate variables; and **B** a metavariable for the formulæ of $\mathscr{L}$. The following are the relevant inductive truth-clauses for the sentences of $\mathscr{L}$, relative to the choices $t_1^*, \ldots, t_n^*; X_1^*, \ldots, X_m^*$; and the assignment Σ :

1. $\mathbf{a} = \mathbf{b}$ is true iff the individual designated by the choice $\mathbf{a}^*$ is identical with the individual designated by the choice $\mathbf{b}^*$, under $\Sigma, t_1^*, \ldots, t_n^*$ for any free predicate variables and terms possibly in **a** and **b**;
2. $\mathbf{a}\eta\mathbf{Y}$ is true iff the individual designated by the choice $\mathbf{a}^*$ is among the individuals designated in the plural choice $\mathbf{Y}^*$, under $\Sigma, t_1^*, \ldots, t_n^*$ for any free predicate variables and terms possibly in **a**;
3. **Pa** is true iff Σ (“**P**”) is a formula ϕ (x) such that the formula resulting from the substitution of “x” by “**a**” $\phi(\mathbf{a}/x)$ is such that $\mathbf{a}^*$, t_1^*, \ldots, $t_n^* \vDash \phi$, for any terms t_1, \ldots, t_n possibly in ϕ;
4. $\exists$ **a B** is true iff, corresponding to the variable **a**, it is possible to perform a singular choice $\mathbf{a}^*$ such that $t_1^*, \ldots, t_n^*; X_1^*, \ldots, X_m^*, \mathbf{a}^*, \Sigma \models \mathbf{B}$;
5. $\exists$ **YB** is true iff, corresponding to the variable **Y**, it is possible to perform a plural choice $\mathbf{Y}^*$ such that $t_1^*, \ldots, t_n^*; X_1^*, \ldots, X_m^*; \mathbf{Y}^*; \Sigma \models \mathbf{B}$;[22]
6. $\exists$**PB** is true iff there is a “**P**”-variant Σ' of Σ such that $t_1^*, \ldots, t_n^*, X_1^*, \ldots, X_m^*, \Sigma' \models \mathbf{B}$.

It has to be kept in mind that acts are not entities, but exactly acts. Recall that ACS is based on Kripke’s intuition of how reference is fixed and works in the natural language. Thus, ideal acts can be conceived as idealisations of actual acts of reference, just as ideal agents can be conceived as idealisations of actual agents. This analogy with actual acts provides a way to make sense of how we can conceive reference to be fixed in formal languages. The quantification on acts in the previous semantic clauses, thus, is to be meant potentially. There is a substantial difference between performed acts and merely potential acts, capable of being performed by the agents. Truth-clauses for singular and plural quantification do not “refer to a totality of acts, conceived as entities existing in a mysterious realm: (…) as acts are

[22] See Martino (2004, pp. 103–133), also for the act of choice clause for the formulæ of the form $\forall$**YB**.

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not entities, it makes no sense to talk of a totality of acts".[23] Thus, the notion of possibility in (3) and (4) implies that, among different potential acts of choice, one, either singular or plural, may be performed such that it verifies **B**. It is absolutely determinate that the agents may perform a simultaneous choice, i.e. a combination of 0, 1, such that it verifies **B**. Thus, the arbitrariness with which 0 or 1 are chosen by each agent does not refute the validity of the Principle of the Excluded Middle. ACS is plainly compatible with classical logic. Although the choice between 0, 1 is arbitrary, it is immediately determinate which the outcomes of any act of choice are. In fact, given some arbitrarily chosen individuals *Y*s and an arbitrarily chosen individual *x*, whether *x* is (or is not) among the *Y*s is an immediate outcome of which individuals *Y*s are chosen.

5 Plural Logicism: The Alternative to Frege

In the present section, I shall explore the philosophical features of PG as an alternative form of logicism.

Frege's logicism is the derivation of arithmetic from logic, where logic is understood as a theory of concepts. To this extent, Frege's Basic Law V plays a philosophically crucial role, since it connects concepts to numbers as extensions.

Clearly, this view is unavailable in PG. Nowadays, what counts as logic is indeed disputed. Regardless of this issue, though, no one would likely commit herself to the claim that logic is a theory of concepts. There is nevertheless agreement on one feature a theory should have in order to be considered as logic, namely ontological innocence. The issue of ontological innocence is rather significant to the debate on the alleged logicality of second-order logic and Boolos' plural quantification.[24] So, I take it that in order to show that PG embodies a form of logicism, PG not only has to derive arithmetic, which in fact it does, but it also has to do it without introducing any unwanted higher-order ontological commitment.

5.1 Ontological Innocence: Comprehension Axioms

First of all, ontological innocence is accomplished through plural quantification. Plural variables, in fact, are interpreted as varying over the first-order domain. Thus, PLC defines pluralities by quantification over pluralities, but this does not lead to problematic impredicativity, because PLC does not introduce a new entity, e.g. the plurality *X*, on the grounds of a totality it belongs to. It just indicates a multiplicity of individuals that we already have at disposal. Plural quantification is just a linguistic tool to talk about those individuals in a way which is not available to regular first-order quantification.

[23] Martino (2004, p. 131), En. trans. mine.

[24] See Linnebo (2003), Parsons (1990), and Resnik (1988), for some criticisms of Boolos' view; and Boccuni et al. (2012) for some criticism of Linnebo, Parsons, and Resnik.

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The notion of plurality, though, has been subject to the criticism that the talk of pluralities is just talk of classes in disguise—or class-like entities.[25] This criticism, nevertheless, assumes tacitly that pluralities are *entities* of some sort, which instead should be firmly rejected. The talk of pluralities is just a *façon de parler*, involving no higher-order entities but only regular first-order individuals plurally considered. ACS shows this clearly, since the notion of plural reference is explained in terms of the notion of simultaneous acts of singular choice. Moreover, acts are not entities, so in ACS there is no hidden ontological commitment other than the first-order.

We can take advantage of the fact that most of the deductive strength of PG comes from plural quantification, and completely eliminate second-order commitment. There are no entities which the predicate letters $F, G, H, \ldots$, refer to, neither classes nor properties nor Fregean concepts. So, at most, what we are really committed to in $\mathscr{L}$, as far as second-order variables are concerned, are the predicates $F, G, H, \ldots$, themselves, considered as linguistically construed entities. As such, only those predicates actually definable in $\mathscr{L}$ are allowed.

To this extent, TIR and VCP* provide a motivation for the *predicative* restriction on PRC: no entity can be referred to only in terms of a quantification over the elements of the domain it belongs to, because there should be a way to refer to it directly, in order not to violate TIR and VCP*. Now, consider that the only way to access a predicate is via language. If so, on the grounds of TIR and VCP*, we are not allowed to specify a predicate only through quantification over predicates, because TIR requires us to be able to exhibit that entity in some way before we present it through a quantification on the domain it belongs to. This requires us to disallow bound predicate variables on the right-hand side of PRC. In this respect, it is rather obvious that substitutional quantification provides the correct semantic framework for predicates, under the assumption of TIR. TIR thus provides also a general, philosophical justification for using two different semantics in $\mathscr{L}$.

On the other hand, given the ontological innocence of plural quantification, the impredicativity involved in PLC is consistent both with TIR and VCP*. For the very same reason, we may also allow free plural variables in PLC. Nevertheless, free plural variables cannot be allowed in the formulæ permissible on the right-hand side of PRC, or in extension-terms for that matter. This is indeed required on pain of contradiction. But I see a further reason for this. Allowing for free plural variables in predicates and extension-terms would amount to having a correspondence between pluralities and singular entities, namely a singular linguistic entity (a predicate) and a single first-order object (the value of an extension-term). Such a correspondence, appropriately regimented, could prove very useful in a set-theoretical setting, where it would provide a motivation for limitation of size as Burgess suggests, or an intuitive starting point for set-formation as Linnebo points out.[26] But in PG, such a correspondence is undesirable, not only on the grounds of Cantor's theorem. In PG, in fact, the extension-terms do not refer to intrinsically set-theoretic objects (see the

[25] See Linnebo (2003), Parsons (1990), and Resnik (1988), for some criticisms of Boolos' view; and Bocculi et al. (2012) for some criticism of Linnebo, Parsons, and Resnik.

[26] I am here referring respectively to J. Burgess "E Pluribus Unum. Plural Logic and Set Theory", and Ø. Linnebo "Pluralities and Sets".

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next Section on this), so a correspondence between pluralities and single entities, though consistently restricted, would sound unmotivated and possibly counterintuitive as for the intuitions we have about pluralities.

5.2 Metaphysical and Ontological Innocence: Extension-Terms and Predicates

In the previous section, I motivated the claim that PG is ontologically innocent as for plural and predicate quantification. In the present section, I will provide motivation for claiming that PG's first-order fragment is *metaphysically* innocent. By the notion of metaphysical innocent, I mean that the first-order fragment of PG interpreted by ACS, though ontologically committed to the existence of infinitely many first-order individuals, is not committed to the existence of individuals with a peculiar nature. In particular, I will claim that the individuals that extension-terms take as values need not be considered as *extensions*, i.e. as intrinsically extensional or even set-theoretical objects, but may be considered as individuals deprived of any intrinsic nature.

Through ACS, the notion of satisfaction is given in terms of arbitrary choices (and substitutional quantification). So, for an individual x to satisfy a formula ϕ means just to be chosen by an arbitrary choice to satisfy ϕ, without appealing to x having the property allegedly expressed by ϕ or being an element of the class allegedly individuated by ϕ. First-order individuals, then, are not conceived as the bearers of properties on the grounds of which they are distinguished from one another. The minimal condition of distinguishability of an individual from another is satisfied through the possibility of choosing and, thus, of naming that very individual instead of another. This general picture provides grounds both for the metaphysical innocence of first-order quantification and the ontological innocence of predicate quantification in PG. Thus, ACS accounts both for the ontological innocence of predicate quantification and the metaphysical innocence of first-order quantification.

Consider, in fact, extension-terms. On Frege's view, extensions owe their logical status to their relation of logical dependence from concepts. In PG, the logical role that Frege assigned to concepts and their relation to extensions are not available. But then again consider that in PG the referents of extension-terms are fixed by ACS.[27] Through ACS, an individual is assigned to the term "$\{x{:}\ \phi\, x\}$" not because that very individual is the extension of all x such that ϕ, rather because such an individual has been arbitrarily chosen as the semantic value of "$\{x{:}\ \phi\, x\}$".[28] Thus, though PG is

[27] Notice how arbitrary reference has been put to use in PG: through it, a whole system of terms, namely extension-terms, is interpreted, unlike Breckenridge and Magidor (2012) where arbitrary reference is considered only as far as particular terms are concerned.

[28] Carrara and Martino (2010) suggests a nominalistic interpretation of the abstraction operator # in Hume's Principle. To this extent, Hume's Principle becomes a full-fledged definition of #. The same goes for {:}, as well as for any other abstraction operator. There is a sense in which this move is unsatisfactory. In order to make sense of the nominalistic interpretation of abstraction principles Carrara and Martino (2010) has to tell a story about what is meant by *abstraction*. The usual meaning is that abstraction introduces or individuates special abstract entities in the domain, e.g. numbers or extensions. Carrara and Martino (2010) rejects this reading. According to Carrara and Martino (2010), through abstraction we abstract from the objects' peculiarities, in order to consider them only under the respect of, e.g. numerosity. Where this may be appropriate for Hume's Principle, it is not clear under which respect we would be considering objects in order to make sense of the extension-operator. In general, the equivalence

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indeed committed to the existence of infinitely many first-order individuals, it is *not* committed to the existence of *intrinsically* extensional objects. For this reason, the Julius Caesar problem is easily solved in PG, since, if Julius Caesar is in PG's domain, then it is capable of being chosen as the semantic value of a singular term in an arbitrary act of choice. So, Julius Caesar may, for instance, play the role of the empty extension, if an ideal agent chooses him to be the semantic value of "0". In this respect, PG's first-order fragment is *metaphysically* innocent, since by ACS it is not committed to a sort of objects that are intrinsically set-theoretical.[29] Axiom V, then, claims a completely arbitrary correspondence between predicates and objects: a certain predicate is not connected to an object because this latter is the extension of all objects that satisfy that predicate, rather it is connected to an object which, once it has been chosen as the semantic value of a given extension-term, *plays the role* of the extension of the objects satisfying the predicate.[30] Such a choice, though, must obey axiom V, in order for ACS to license models of PG. In fact, it might be argued that, since acts of choice are arbitrary, nothing would prevent agents from picking the same individual with respect to two extension-terms, even though the formulæ in those terms were not equivalent (or the other way around, for that matter).

Footnote 28 continued
relation on the right-hand side of any abstraction principle should provide a specific respect under which to perform abstraction. I doubt this is achieved: are identity of directions or equivalence of concepts enough fine-grained? Moreover, in Hume's Principle or Basic Law V as definitions, the first-order variables on the right-hand side of the biconditional should not take numbers or extensions as values, because otherwise we would need a way to individuate them before we stipulate the definition of, respectively, # or {:}. Nevertheless, first-order impredicativity in abstraction principles is also their strength. I thus prefer to hold the usual reading of abstraction operators: they individuate objects. Still, these objects are not, in my view, intrinsically numbers or extensions.

[29] It has to be stressed that PG's first-order fragment is not *ontologically* innocent, since it proves the existence of infinitely many individuals. This may cast doubts upon the logicality of PG, especially if one holds the view that logic does not imply any ontological commitment at all. Consider, for instance, Boolos' view: "In logic we ban the empty domain as a concession to technical convenience but draw the line there: We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. After all, logical truth is just *truth no matter what things we may be talking about and no matter what our (nonlogical) words mean*. Since there might be fewer than two items that we happen to be talking about, we cannot even take $\exists x \exists y (x \neq y)$ to be valid." (G. Boolos, "The Consistency of Frege's *Foundations of Arithmetic*", in G. Boolos, *Logic, Logic, and Logic*, p. 199). Nevertheless, it may be the case the one holds a different view of logic, as Frege and the neo-Fregeans do, according to which logic is a theory about some special kind of truths with its own special objects, namely logical objects. So, after all, whether PG's first-order fragment is logic or not depends upon the view of logic one subscribes to. Regarding Boolos' previous quotation, in fact, Wehmeier (1999, p. 326) points out: "This argument is so anachronistic that it seems quite unsatisfactory to me: Evidently Frege *wanted* his theory to prove the existence of infinitely many objects and still conceived of it as logical. And what if there really *are* infinitely many logical objects—why should logic not prove their existence? Be that as it may, one might argue that the provability of the existence of infinitely many objects other than logical ones is a *reductio ad absurdum* of a logicist system." If one holds on to a Fregean view of logic, then she may consider PG as an alternative form of logicism. Frege's opponent simply will not, and that is fair enough. Of course, the issue posed by Wehmeier about the existence of non-logical objects requires a detailed argument on why we should not be bothered by it. This issue, though, can not be investigated in this paper, not at the length it deserves. So, I shall leave it open for further analysis.

[30] See Breckenridge and Magidor (2012), where arbitrary reference is taken into account only as for instantial reasoning, namely for particular terms. In PG, on the other hand, ACS provides a way to apply arbitrary reference to a whole system of terms, namely extension-terms.

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Nevertheless, axiom V puts a restraint on the models ACS delivers: with respect to two extension-terms "$\{x\colon \phi\ x\}$" and "$\{x\colon \psi\ x\}$", agents pick the same arbitrary individual under $\Sigma, t_1^*, \ldots, t_n^*$ if, and only if, $\phi\ x$ and $\psi\ x$ turn out to be equivalent, under $\Sigma, x^*, t_1^*, \ldots, t_n^*$.

Given that in PG concepts play no role at all, an obvious question concerns why not to obliterate predicates and work exclusively with pluralities all along, having only PLC and axiom V interacting. Nevertheless, predicates do play an important part in PG, a part that would seem inappropriate for pluralities to play. Their essentiality lies in that we may have extension-terms corresponding to every predicate, as in fact the formulæ permitted in extension-terms are all the PRC-permissible formulæ, so to every such formula there corresponds a predicate which is the value of a variable F.[31] If we had only PLC and axiom V, axiom V should be restricted consequently in order to avoid inconsistency: not to every plurality, in fact, there might correspond an extension, on pain of contradiction. For instance, axiom V could be restricted as not to allow for free plural variables. In this way, inconsistency would be avoided. Nevertheless, this latter would be the only reason to regiment the interaction between pluralities and extensions. The restriction would consequently sound quite *ad hoc*. On the other hand, the predicative restriction on predicates in extension-terms not only avoids inconsistency, but it also is motivated by TIR and VCP* on the grounds of the linguistic nature of predicates. I mentioned before that we cannot define a predicate merely on the grounds of a quantification on the domain of predicates, *pace* the Platonist: since predicates cannot be defined through formulæ containing bound predicate variables, no extension-terms correspond to those formulæ. In this perspective, the restriction connected with using bound predicate variables in extension-terms is motivated by the justification for the restriction concerning bound predicate variables in PRC. Predicates function as mediums between PLC and axiom V: they filter pluralities. The restrictions on PRC provide a criterion to distinguish between which pluralities form extension-terms and which do not.

It may be further objected, though, that predicates can be expunged from PG, and first-order formulæ alone can be consistently allowed in extension-terms. This would provide the resulting system, call it PG′, with infinitely many individuals, thanks to first-order axiom V, and full second-order induction, thanks to unrestricted PLC. As a result, PA^2 would be interpretable in PG′. I see the following reason not to take this suggestion either. Though it is quite likely that both PG and PG′ are equi-consistent with PA^2, PG is slightly more expressive than PG′. In PG, unlike PG′, ω can be explicitly defined, and in general in PG there are extension-terms that cannot be defined in the language of PG′. Given the formulæ permitted in PRC, extension-terms in PG may be defined by Σ^1_1-formulæ of the form $\exists X \ldots X \ldots$. PG provides the means to express more theorems about the universe of discourse than PG′.[32]

[31] Possibly, using λ-notation would be more transparent. Though, this would unnecessarily complicate the notation, but it surely is a way to go if some confusion should arise.

[32] See Boccuni (2011b) for some similar considerations.

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6 Conclusion

In the present article, I presented the predicative second-order system PG, which interprets second-order Peano arithmetic (Sects. 1 and 2). The main features of PG are two different kinds of second-order quantification, namely predicate quantification and plural quantification, an appropriately restricted formulation of Basic Law V, and ACS by Martino.

ACS is motivated starting from some independent considerations about arbitrary reference in mathematical and logical reasoning. The two main issues concerning arbitrary reference are its genuine referentiality and the relation of logical presupposition that quantification bears to it (Sects. 3 and 4). The very notion of arbitrary reference is then applied to PG, in particular to extension-terms (Sect. 4), providing first-order metaphysical innocence, and a possible new approach to the Julius Caesar problem (Sect. 5.2)

At the same time, plural quantification and substitutional quantification, which interprets predicate quantification, contribute to accomplishing second-order ontological innocence (Sect. 5.1)

I finally claimed that, on the grounds of ACS and substitutional quantification, PG embodies a form of logicism which is radically different from Frege's, as it is grounded on the existence of individuals and their metaphysical neutrality, rather than on the existence of concepts.

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