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On the Consistency of a Plural Theory of Frege's *Grundgesetze*

Abstract. PG (Plural *Grundgesetze*) is a predicative monadic second-order system which is aimed to derive second-order Peano arithmetic. It exploits the notion of plural quantification and a few Fregean devices, among which the infamous Basic Law V. In this paper, a model-theoretical consistency proof for the system PG is provided.

Keywords: Plural logic; Basic Law V; Consistency; Second-Order Peano Arithmetic

Overview

The aim of the paper is to provide a model-theoretic consistency proof for the predicative second-order system PG, which employs plural quantification and a formulation of Frege's infamous Basic Law V. This system is of foundational interest because, unlike the consistent fragments of Frege's *Grundgesetze* so far provided,[1] it interprets second-order Peano arithmetic, as explained in [2]. The consistency proof is crucially based on [8].

1. The Theory PG

The language $\mathscr{L}$ of the theory PG consists of

(i) an infinite list of singular individual variables $x, y, z, \ldots$;

(ii) an infinite list of plural individual variables $X, Y, Z, \ldots$, that vary *plurally* over the individuals of the first-order domain;

(iii) an infinite list of monadic concept variables $F, G, H, \ldots$;

(iv) the logical constants $\neg$, $\rightarrow$, $=$;

(v) existential quantifiers $\exists$ for every kind of variables;

(vi) the constant η;

(vii) the abstraction function $\{:\}$.

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[1] See [7], [8], and [13].

The atomic formulæ of $\mathscr{L}$ are:

(viii) $\mathbf{a} = \mathbf{b}$;

(ix) $\mathbf{a}\eta\mathbf{Y}$;[2]

(x) $\mathbf{Pa}$,

where $\mathbf{a}$ and $\mathbf{b}$ are metavariables for the terms of $\mathscr{L}$, $\mathbf{Y}$ is a metavariable for plural variables and $\mathbf{P}$ is a metavariable for concept variables. Formulæ of kind (ix) express what I may call *plural reference*, meanwhile formulæ of kind (x) express regular predication. Primitive existential quantification for every kind of variables is available. Universal quantification for every kind of variables can be defined in the obvious manner. The semantic clauses for the formulæ of $\mathscr{L}$ will not be provided here.[3] It is worth mentioning, though, that plural quantification is meant to be interpreted by *Boolos's plural semantics* as in [3], whereas second-order quantification is interpreted as varying over a domain of (Fregean) concepts.

Together with the singular variables *x, y, z, . . .*, the *terms* of $\mathscr{L}$ are:

(xi) an infinite list of extension-terms of the form $\{x : \psi(x)\}$,

where $\psi(x)$ is a formula of $\mathscr{L}$ containing neither bound concept variables nor free plural variables, with x as the designated variable. It may contain, though, free concept variables, bound plural variables, and both free and bound singular variables. Also, nested extensions may appear in extension-terms.

Two Comprehension Principles are provided: a *Plural Comprehension Principle*

PLC $(\exists X)(\forall y)(y\eta X \leftrightarrow \phi(x))$,

where $\phi(x)$ does not contain X free; and a *Predicative Comprehension Principle*

[PRC] $(\exists F)(\forall x)(Fx \leftrightarrow \psi(x))$,

where $\psi(x)$ contains neither F free, nor free plural variables, nor bound concept variables. However, $\psi(x)$ may contain both free concept variables and bound plural variables. A schematic formulation of *Basic Law V* is also among the axioms:

[V] $\{x : \psi(x)\} = \{x : \chi(x)\} \leftrightarrow \forall x(\psi(x) \leftrightarrow \chi(x))$.

Clearly, $\psi(x)$ and $\chi(x)$ are subject to the restrictions mentioned earlier concerning extension-terms. Axiom V guarantees the existence of Dedekind-infinitely many first-order individuals in the domain. This is crucial to guarantee that Peano axioms may be derived in PG.

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[2] To be read '**a** is among the **Y**s'.

[3] Detailed semantics for $\mathscr{L}$ shall be found in [2].

It is worth noticing that the restrictions on the right-hand side formula of PRC are exactly the same restrictions imposed on the formulæ permitted in the extension-terms. This guarantees that in PG, for every concept, there is the corresponding extension and that every extension is defined by a concept.[4]

Notice that, strictly speaking, $\mathscr{L}$ is monadic. Nevertheless, terms like (x, y) for ordered pairs can be introduced by the usual Wiener-Kuratowski definition and they will have denotations assigned already in Section 4.1.1: in fact, both the notions of singleton and unordered pair are definable in $\mathscr{L}$. Thus, $\mathscr{L}$ is provided with polyadic expressive capacity.[5]

In PG, natural numbers may be inductively defined as extension-terms, as

Definition 1. $0 =_{df} \{x : x \neq x\}$;

Definition 2. $1 =_{df} \{x : x = 0\}$;

Definition 3. $2 =_{df} \{x : x = 1\}$;

and so on. In general, the successor of a number is its singleton. A plurality X is said to be *inductive* whenever it contains 0 and is closed under successor. That such a plurality exists is guaranteed by PLC, which the formula $x\eta X \leftrightarrow \forall Y(0\eta Y \wedge (x\eta Y \rightarrow \{x\}\eta Y))$ is a valid instance of. The usual definition of the set of natural numbers may be given in terms of pluralities. First, a concept $\mathbb{N}$ is defined and, secondly, the corresponding extension ω is introduced:

Definition 4. $\mathbb{N}x =_{df} (\forall Y)(Y \text{ is inductive} \rightarrow x\eta Y)$;

Definition 5. $\omega =_{df} \{x : \mathbb{N}x\}$.

The following formulations of second-order Peano axioms are derivable in PG, where the singular variables x and y are restricted to ω:

Theorem 6. $\mathbb{N}0$*;*

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[4]This is slightly different from Heck's predicative system, as Heck imposes no restrictions on extension-terms and consequently on Basic Law V. Nevertheless, if unrestricted Basic Law V were admitted in PG, the resulting system PG* would be inconsistent. This is easily seen by introducing a plurality X defined by the instance $x\eta X \leftrightarrow x \notin x$ of PLC and noticing that any resulting extension $\{x : x\eta X\}$ could be used to derive a plural version of Russell's paradox, a definition of membership *á la* Frege being easily provided as $x \in y =_{df} \exists F(y = \{z : Fz\} \wedge Fx)$.

[5]See [2].

Theorem 7. $(\forall x)(\{x\} \neq 0)$*;*

Theorem 8. $(\forall xy)(y = \{x\})$*;*

Theorem 9. $(\forall xy)(\{x\} = \{y\} \to x = y)$*;*

Theorem 10. $(\forall X)(0\eta X \wedge (\forall x)(x\eta X \to \{x\}\eta X) \to (\forall x)(x\eta X))$.[6]

PG is significantly stronger both than [7], [8], and [13], as these fragments of Frege's *Grundgesetze* are very unlikely to interpret second-order Peano arithmetic.[7] This latter fact seems to put a serious limitation on the recovery of Frege's original programme. Nonetheless, that PG is sufficiently strong to derive (the plural version of) second-order Peano axioms seems to suggest that, to some extent, Frege's programme is not doomed to failure after all. In this perspective, though, the philosophical issues concerning whether this present system is *logicistically* acceptable are very important.[8]

2. Plural Logic vs Second-Order Logic

It may sound unsurprising that second-order Peano arithmetic is interpretable in PG. Any theory that both guarantees Dedekind-infinitely many individuals and employs full second-order comprehension will do it, after all. And, clearly, PG gets the former from Basic Law V and the latter from PLC. Nevertheless, leaving aside the exact mathematical strength of PG,[9] this result is not trivial on the grounds of the threat of inconsistency, to the extent

[6] For further details on PG, see [2].

[7] See [4], Sections 2.1 and 2.6.

[8] See [2].

[9] I claim that PG is at least as strong as third-order Peano arithmetic. Consider, first, that second-order Peano arithmetic is equi-interpretable with ZF^-, i.e. first-order Zermelo set-theory without the axioms of power-set, replacement, and choice; and third-order Peano arithmetic is equi-interpretable with $ZF^- + \wp(\omega)$, where the power-set axiom is replaced by a weaker axiom stating the existence of the power-set of ω. On this, see [4] 70–71. Now, since PG interprets second-order Peano arithmetic, PG clearly is at least as strong as ZF^-. Consider now that in PG pluralities may mimic functions. Under the definitions both of natural numbers as Von Neumann ordinals and exponentiation x^y as the extension of all pluralities of ordered pairs whose first element belongs to y and the second to x, in PG we may define the extension 2^ω as the extension of all pluralities of ordered pairs whose first element is a natural number and the second is either 0 or 1, for $2 =_{df} \{0, 1\}$. Under the assumption of classical logic, this is equivalent to the existence of the power-set of ω, where each subset of ω corresponds uniquely to a plurality of ordered pairs whose first element z belongs to ω and the second is 1 if, and only if, $z \in \omega$; and 0 otherwise. This does not seem to be available in Burgess's theory, on the grounds that neither ω is definable in it nor bound concept variables - neither predicative nor impredicative - are

that the problematic assumptions leading Frege's *Grundgesetze* to contradiction, i.e. (i) full second-order logic and (ii) the existence of an extension for every concept, are incorporated in PLC and Basic Law V respectively.

Some found a way to bring (i) and (ii) together consistently, through the introduction of a 'safety zone' between the defining power of full second-order logic and the existential assumption concerning extensions, which on the one hand avoids the inconsistency and, on the other, prevents the resulting system from too narrow a regimentation. Such a solution is, for instance, proposed by [4]. It consists of a three-sorted second-order system with a round of second-order variables $X, Y, Z, \ldots$, varying over concepts in general,[10] and a further round of second-order variables $X^0, Y^0, Z^0, \ldots$, varying over 'simple' concepts. Burgess tries to implement the simplicity requirement modelled on Russell's intuition that only 'simple' propositional functions have extensions. 'Simple' may be taken to mean 'predicative'. Burgess's theory, thus, contains: a full impredicative comprehension axiom for general concepts [Impr-CA] $(\exists X)(\forall x)(Xx \leftrightarrow \phi(x))$,

where $\phi(x)$ does not contain X free; a restricted comprehension axiom for simple concepts

[Pred-CA] $(\exists X^0)(\forall x)(X^0 x \leftrightarrow \psi(x))$,

where $\psi(x)$ contains neither X^0 free nor bound simple concept variables *nor bound nor free general concept variables.*[11] Furthermore, Basic Law V is among the axioms, where only 'simple' concepts have extensions.[12] The

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allowed in extension-terms. PG would thus be at least as strong as $\mathrm{ZF}^- + \wp(\omega)$, and consequently at least as strong as third-order Peano arithmetic. Nevertheless, what the exact mathematical strength of PG is, though a very intriguing issue, is material for a further article. Remarks on this topic may be found in [2].

[10] I'll call these variables *general* concept variables from now on, not to mean that they vary over some special kind of concepts - general concepts - but rather to recall that they vary over concepts indiscriminately.

[11] This theory is Heck's predicative second-order fragment of *Grundgesetze* (PV), as presented in [8], augmented with second-order logic (SOL), and it is equi-consistent with second-order Peano arithmetic. Burgess does not explicitly mention the restrictions on Pred-CA concerning general concept variables, but these are indeed necessary, since otherwise paradox would be derivable, as the domain of simple concepts is a subdomain of the domain of general concepts.

[12] See [4] 119: 'An alternative approach would be to have, in addition to variables $X, Y, Z, \ldots$ for concepts in general, special variables $X^0, Y^0, Z^0, \ldots$ for 'simple' ones, with a restricted version of comprehension for these, and only these allowed to have extensions. (...) Again if 'simple' is understood as predicative, we will get something familiar, PF, but again with full, impredicative second-order logic', where PF is predicative Frege set theory, i.e. a first-order set theory with a predicative axiom of separation and the axiom of extensionality. This latter is easily obtained from PV. See [4] 87–91.

introduction of predicative comprehension and the consequent restrictions on Basic Law V avoid the inconsistency, only if impredicative second-order quantification does not interact with extensions at all. This sounds like throwing out the baby with the bath water. [4], in fact, interprets second-order Peano arithmetic, as augmenting PV with SOL provides enough deductive capacity to recover full second-order induction, but it requires such restrictions in order to avoid inconsistency that it is not able to derive desirable theorems, like the one asserting the existence of the extension of all natural numbers.

PG, unlike [4], operates with two kinds of second-order variables whose domains are distinct, rather than one a subdomain of the other. This feature of PG is crucial as it allows concepts to be defined from full second-order (plural) quantification in PRC. So, to some extent, through Basic Law V, impredicative second-order logic is interacting with the existential assumption (ii). This equips PG with a strength that is not available to Burgess's system, but at the same time clearly demands for a consistency proof.

As for PG's mathematical strength, in fact, notice that the restrictions on PRC do not prevent PG from proving interesting theorems like the existence of the extension of the concept of Number. The extension containing all natural numbers, in fact, may be explicitly defined in PG, whereas it cannot be in [4].[13] In general, in the language of PG there are extension-terms which cannot be in Burgess's language,[14] given the restrictions that need to be imposed on this latter. Given the formulæ permitted in PRC, terms containing Σ^1_1-formulæ of the form $\{x : (\exists X)(x\eta X)\}$ are allowed in PG, beside the Σ^0_1-terms $\{x : Fx\}$. These Σ^1_1-terms need to be assigned denotations. PG, thus, is not logically identical with a predicative second-order system bluntly augmented with full second-order logic, rather it is significantly stronger. For this reason, there is no obvious way to prove PG's consistency through equi-consistency with [4] or any other similar system, but PG's consistency should be provided through some other means.

As for the consistency demand, notice that the model here presented is quite intuitive. Nevertheless, its importance lies exactly in the fact that its existence proofs PG's consistency. PG, in fact, lets the two problematic assumptions (i) and (ii) interact with each other to the extent that pluralities interact with extensions through plural quantification. Consequently, a proof is needed that this interaction is safe.

[13]See [4]. Nevertheless, the general concept of Number $\mathbb{N}$ may be defined through ImprCA. This is the reason why second-order Peano arithmetic is interpretable in PV+SOL.

[14]See [4].

Furthermore, the fact that PG employs both predicative and full plural comprehension possibly makes this set up rather unfamiliar. For all these reasons, the demand for a consistency proof is urgent.

3. Philosophical Remarks

Several philosophical issues may arise concerning PG. In what follows, I'll try to address some of them, partly relying on [2] where I take some of those issues into closer consideration.

The overall significance of augmenting a consistent predicative fragment of Frege's *Grundgesetze* with plural logic lies both in the mathematical strength this latter provides to that fragment and in the philosophical motivations for its application to it. On the one hand, in fact, this result challenges the claim that, given the mathematical strength of the consistent fragments mentioned in the introduction, Frege's logicism is mathematically unfeasible. On the other, the philosophical motivations underlying PG should be provided. These are part of a more extensive project of revising Frege's logicism through plural logic,[15] but some of them shall be investigated in the remainder of this paragraph.

Prima facie, plural variables and second-order variables behave very similarly: from a logical point of view, they are both second-order variables, governed by appropriate comprehension axioms. Nevertheless, regular predication of the form $\mathbf{Pa}$ attributes properties to individuals, whereas plural reference of the form $\mathbf{a}\eta\mathbf{X}$ just provides means to consider those individuals plurally. In this latter case, we do not assume the existence of an entity, e.g. the plurality X, containing some first-order individuals, rather we employ a linguistic tool, i.e. plural quantification, in order to talk of first-order individuals in a way which is not available to usual first-order logic. As for first-order and plural quantification, we do not maintain singular reference, on the one hand, while multiplying the sorts of entities, i.e. regular first-order individuals as opposed to pluralities, on the other; rather, we multiply the sorts of reference, i.e. singular and plural, while maintaining just one sort of entities, i.e. first-order individuals. Plural variables and second-order variables may be taken to do approximately the same job logically,[16] but not

[15] See [2], in particular as far as a motivation for predicativism about concepts is concerned.

[16] This may be disputed, though, on the grounds of some results of mathematical strength. Though plural logic and second-order logic are inter-definable, the argument from the previous section seems to show that at least some applications of those two logics may lead to different mathematical results. So, to some extent, plural logic and

ontologically. Thus, one reason for allowing both in PG is that we can take advantage of full second-order definability and have it interacting safely with Basic Law V, without unwanted ontological commitment. Another reason for having both kinds of quantification is that having usual second-order variables allows us to hold the Fregean assumption that extensions are defined by concepts, since in PG's intended interpretation second-order variables vary over a domain of concepts.[17] Clearly, this assumption is governed by the restrictions respectively on extension-terms and on the PRC-permissible formulæ. This, together both with the mathematical strength plural logic is supposed to provide to PG and its ontological parsimony, may be of some interest to the Fregean. Notice that this is not to claim that Frege himself would be happy with a distinction between concepts and pluralities. There are indeed several arguments that he possibly would not.[18] Still, anyone engaged in the enterprise of vindicating Basic Law V from unjust ostracism (for instance on the grounds of the so-called 'bad company problem', or inconsistency) may find these results interesting.[19]

A further issue concerns whether we have linguistic evidences from natural language that we are allowed to use both second-order and plural predication. This is a rather delicate topic: different authors have very different conceptions of the roles plurals and predicates play in natural language, and their possible interactions. It cannot be rejected, though, that plurals are in systematic use of natural language and they were so, far before we started to speculate about Fregean concepts. On the other hand, we also use predicative phrases to attribute properties to individuals. [1] provides some arguments that in natural language we distinguish between attributive and predicative expressions, which may be translated respectively into regular and plural predication:

> Many adjectives are used attibutively (in the philosophical sense), i.e., what they attribute to a particular of which they are predicated depends on its classification. A good lecture is not good in the same way as someone's eyesight may be good; a big mouse is not the size of a big elephant; and so on. (...) Attributive adjectives cannot

second-order logic do not do the same job even logically.

[17]See also [2], [7], and [13].

[18]See, for instance, [6] 93, on Frege's logical analysis of sentences as constituted by concept-phrases and arguments: 'There is no such thing as a "plurality", which is the misbegotten invention of a faulty logic: it is only as referring to a concept that a plural phrase can be understood ... But to say that it refers to a concept is to say that, under correct analysis, the phrase is seen to figure predicatively.'

[19]See, for instance, [10] and [11].

> easily be used as referring expressions. The property they attribute depends on the presence of a noun which classifies the particulars to which they attribute the property. Consequently, if they were used as referring expressions, without any other concept determining the property by which they are to pick out the particulars referred to, it would be indeterminate which property is supposed to determine their reference. Thus, if they are to be used to refer to particulars, a noun that specifies which kind of particulars are talked about should be presupposed. (...) On the other hand, some adjectives are used predicatively, i.e., the property they attribute is independent of the classification of the thing to which they attribute it. Such adjectives can be used as referring expressions. The things they will then denote are those that have the property that they attribute when used as adjectives.[20]

Consider, for instance, the adjective 'little', as in

(1) Children are little.

In order to use 'little' as a referring expression in a sentence, we should accompany it with a noun which classifies the sort of individuals to which we are ascribing the property of being little. So, instead of saying

(2) The littles will arrive late,

we should be saying

(3) The little brothers/dogs/ones will arrive late,

where the use of 'one', though, is either anaphoric on some previous specification, like in 'Silly people may be fun, but clever ones are funnier', or is colloquial, e.g. the audience knows who has been talked about.[21] So, attributive adjectives are used (only or largely) in a conceptual sense, i.e. as ascribing properties to individuals.

Consider now the adjective 'square'. It may be used both into predicate position, i.e. attributively, as in

(4) Office tables are square;

and as a referring (plural) expression, as in

[20] [1] 33–34.

[21] See [1] 33.

(5) My kitchen floor is tiled in large squares of black and white marble.

In (4), I am ascribing the property of having a certain shape to office tables, so I am using 'square' attributively; whereas in (5) I am using 'square' as a noun to refer (plurally) to some objects my kitchen floor is made of.[22] To be sure, I would not be allowed to say 'My kitchen floor is tiled in *larges* of black and white marble'. [1] suggests an account of these features of natural language which is grounded on plural reference and plural logic, and provides motivations for using both plural logic and usual second-order logic side by side.

A further issue concerns whether there are any independent considerations that shape the restrictions imposed on the formulæ permissible on the right-hand side of PRC. As for bound second-order variables, in [2] a justification for disallowing them in PRC is provided, on the grounds of [5] and [12]. According to [5] and [12], the possibility of directly referring, at least ideally, to any object of a Universe of Discourse is presupposed both by logical and mathematical reasoning, also when non-denumerable domains are concerned. As a consequence, quantification itself logically presupposes the ideal possibility of directly referring to each and every element of a domain, before we consider those elements through generalisation. [5] and [12] call this claim the *Thesis of Ideal Reference* (TIR). From this perspective, reference to an entity exclusively in terms of a quantification on the domain it belongs to cannot be allowed, because it is required that we are able to directly refer to that entity, even if just in an ideal way, on pain of violating TIR. As a corollary of TIR, in fact, Martino provides a re-formulation of Russell's well-known *Vicious Circle Principle*

VCP* No universe of discourse can contain an element which we can refer to only through quantification over that universe.[23]

TIR and VCP* provide grounded reasons for holding a predicativist stance as far as the intended interpretation of second-order quantification is over *intensional* entities like Fregean concepts, properties, or predicates. In fact, the only way to access an intensional entity is via language, i.e. through a formula which expresses it. Now, consider the impredicative comprehension principle in second-order logic: $\exists F \forall x(Fx \leftrightarrow \phi(x))$. Considering the concept F means considering the formula $\phi(x)$, under its intended interpretation. Therefore, $\phi(x)$ cannot contain any bound second-order variable

[22] All these examples are taken or paraphrased from [1] 33–34.

[23] Notice that VCP* follows from TIR also when non-denumerable domains are concerned.

varying over intensional entities, on pain of violating TIR and VCP*. This clearly requires us to put some restrictions on comprehension for second-order intensional entities.[24] On the other hand, the impredicativity in PLC is consistent both with TIR and VCP*. In fact, PLC does not impredicatively introduce a new entity, e.g. the plurality X. Rather, it just indicates a multiplicity of individuals that we already have at disposal and, as such, these individuals are, at least in principle, capable of being referred to before quantification is used to take them into account, as TIR and VCP* require. Plural quantification is just a linguistic tool to talk about those individuals in a way which is not available to regular first-order quantification.[25]
At the same time, for this very latter reason, free plural variables cannot be allowed in the formulæ permissible on right-hand side of PRC, since using them in PRC - or in extension-terms, for that matters - would amount to reify pluralities and identify them with a singular entity, i.e. the concept or the extension such-and-such.

4. Consistency

In order to show that $\mathscr{L}$ has a model, an interpretation for the free variables has to be fixed and the extension-terms have to be assigned denotations. The domains for singular, concept, and plural variables will be provided in the following paragraphs. As for the assignment of denotations to extension-terms, recall that the formulæ allowed in extension-terms are subject to restrictions. In particular, they may contain free predicate variables, bound plural variables, and bound and free singular variables, but they can contain neither free plural variables nor bound predicate variables. On the grounds of these restrictions, I shall first assign denotations to the extension-terms containing the formulæ $\mathbf{A}_i(x)$, which contain neither concept variables at all, neither free nor bound, nor free plural variables, but may contain bound plural variables. Secondly, I will get to assign denotations to the extension-terms containing the formulæ $\mathbf{B}_i(x)$, which contain only free concept variables as second-order variables. Both $\mathbf{A}_i(x)$ and $\mathbf{B}_i(x)$ may contain bound and free singular variables. Nevertheless, these latter are eliminated via substitution by numerals. All this shall suffice to assign denotations to all extension-terms of $\mathscr{L}$.

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[24]See [2].

[25]For the very same reason, we may also allow free plural variables in PLC.

4.1. Domains for Individual Variables

For *individual variables* I mean both singular individual variables x, y, z, $\ldots$, and plural individual variables $X, Y, Z, \ldots$ I shall assume that the constants of $\mathscr{L}$ are denumerably many numerals $\overline{n}$ each of which denotes the corresponding natural number n, so that the first-order domain is the set of natural numbers $\mathbb{N}$. If this extended theory is consistent, so will be its subtheory PG. The assumption on numerals will cover the interpretation of free singular variables in the proof of consistency. Furthermore, let the domain for plural variables be the power-set of $\mathbb{N}$, $\wp(\mathbb{N})$. So a set α belongs to $\wp(\mathbb{N})$ if and only if it is a subset of $\mathbb{N}$.

4.1.1. Extension-Terms Not Containing Free Concept Variables

In this Section, I will assign denotations to the extension-terms containing the formulæ $\mathbf{A}_i(x)$.

Define the *rank* of an extension-term as follows: if $\mathbf{A}_i(x)$ contains no extension-term at all, then the rank of $\{x : \mathbf{A}_i(x)\}$ is 0. If $\mathbf{A}_i(x)$ contains some extension-term and the greatest rank of the terms contained in it is n, then $\{x : \mathbf{A}_i(x)\}$ is rank $n+1$.

Order all terms in a $\omega \times \omega$ sequence, where the terms of each rank form an ω sequence and, for any extension-term t, each term preceding t is of rank less than or equal to the rank of t. Let $J(m, n)$ be a pairing function that assigns a natural number to every ordered couple of natural numbers (m, n) and define the function $J^0(m, n)$ as follows: $J^0(m, n) =_{df} 2J(m, n)$.

By induction, it has to be shown that, the first term in the sequence has a denotation and, for any term t, if all the terms preceding t have denotations then t has a denotation.

The inductive basis can be stipulated: let $\{x : x \neq x\}$ be the first term of the sequence and assign the number $J^0(0, 0)$ to it as its denotation.

Let t be some term in the ordering and, by induction hypothesis, assume that we have assigned denotations to every term prior to t.

Assume moreover that, if the terms $\{x : \mathbf{A}_k(x)\}$ and $\{x : \mathbf{A}_m(x)\}$ precede t in the ordering, then they have the same denotation if, and only if, $\mathbf{A}_k(x)$ and $\mathbf{A}_m(x)$ are equivalent under the interpretation of individual variables described at the beginning of 4.1, if there are any in $\mathbf{A}_k(x)$ and $\mathbf{A}_m(x)$ (i.e. for every term n, $\mathbf{A}_k(\overline{n})$ is true if and only if $\mathbf{A}_m(\overline{n})$ is, under $\mathbf{I}$).

Let t be $\{x : \mathbf{A}_i(x)\}$. Assign the denotation of the term u to it as its denotation, if u is some prior term $\{x : \mathbf{A}_j(x)\}$ such that $\mathbf{A}_j(x)$ is equivalent to $\mathbf{A}_i(x)$. If there is no such term, assign the number $J^0(m, n)$ to t as its denotation, such that m is the rank of t and n is the smallest number such that $J^0(m,k)$ has not yet been assigned as denotation to some extension-term (basically, $J^0(m,k)$ should be still available in the sequence).

Check now that, if $\{x : \mathbf{A}_k(x)\}$ and $\{x : \mathbf{A}_m(x)\}$ precede or are identical with t, then $\mathbf{A}_k(x)$ and $\mathbf{A}_m(x)$ are equivalent if, and only if, $\{x : \mathbf{A}_k(x)\}$ and $\{x : \mathbf{A}_m(x)\}$ have been assigned the same denotation. The case where they precede t has been covered already by the previous assumptions. Now assume $t = \{x : \mathbf{A}_k(x)\}$, thus the denotations of the terms $\{x : \mathbf{A}_i(x)\}$ and $\{x : \mathbf{A}_k(x)\}$ are identical. Then, $\mathbf{A}_i(x)$ is equivalent to $\mathbf{A}_m(x)$ if and only if $\{x : \mathbf{A}_i(x)\}$ and $\{x : \mathbf{A}_m(x)\}$ are assigned the same denotation, by construction through the function $J^0(m,n)$.

4.2. Domain for Concept Variables

Let $\pi(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$ definable by an arbitrary PRC-permissible formula $\psi(x)$, that is, the set β such that, for all $n \in \mathbb{N}$, $\psi(\overline{n})$ is true if, and only if, $n \in \beta$. The formula $\psi(x)$ contains neither bound concept variables nor free plural variables. However, it may contain free concept variables and bound plural variables. Obviously, this construction ensures that $\pi(\mathbb{N}) \subseteq \wp(\mathbb{N})$.

4.2.1. Extension-Terms Containing Free Concept Variables

In this Section, I will assign denotations to the extension-terms containing formulæ $\mathbf{B}_i(x)$. I shall take advantage of the following fact: given an interpretation $\mathbf{I}$ of the free variables in a formula $\psi(x)$ of $\mathscr{L}$, there is a formula $\psi'(x)$ containing no free concept variables at all and no bound plural variables not contained in $\psi(x)$ already, which is equivalent to $\psi(x)$ relative to the values $\mathbf{I}$ assigns to the free variables in $\psi(x)$ and in $\psi'(x)$.[26]

For any formula $\mathbf{B}_i(x)$, there is a formula $\mathbf{A}_i(x)$ equivalent to $\mathbf{B}_i(x)$ under $\mathbf{I}$ relative to x, such that it contains no free concept variables nor bound plural variables at all. In fact, fix an interpretation $\mathbf{I}$ and consider a formula $\mathbf{B}_j(x)$ that contains $V_1, \dots, V_n$ free concept variables. Let $\beta_k \in \pi(\mathbb{N})$ be the set assigned to V_k by $\mathbf{I}$. There are, then, formulæ $\beta_k(x)$, containing no free concept variables and no bound plural variables at all, whose extensions are the $\beta_k \in \pi(\mathbb{N})$. Let $\mathbf{A}_j(x)$ be the formula resulting from the substitution of the V_k by the $\beta_k(x)$ in $\mathbf{B}_j(x)$. Then, $\mathbf{A}_j(x)$ is equivalent to $\mathbf{B}_j(x)$ under $\mathbf{I}$ relative to x. If, then, $\{x : \mathbf{B}_j(x)\}$ is the extension-term corresponding to

[26] In short, under $\mathbf{I}$ relative to x, as x is the designated variable both in $\psi(x)$ and $\psi'(x)$.

$\mathbf{B}_j(x)$, assign the denotation of $\{x : \mathbf{A}_j(x)\}$ to it as its denotation under $\mathbf{I}$. The denotation of the term $\{x : \mathbf{A}_j(x)\}$ has been fixed in Section 4.1.1.

Consider extension-terms containing free concept variables and n extension-terms nested in them. The nested extension-terms may contain free concept variables, bound plural variables and extension-terms too. Notice that the least nested extension-term has been assigned a denotation already as in the preceding paragraphs. So, if all the extension-terms already assigned denotations are substituted by their respective denotations (the numerals for their denotations), the overall extension-term will contain free concept variables as the only free variables and no extension-terms at all. These terms have been assigned denotations in the previous paragraph.

4.3. Every Instance of Basic Law V is True in This Model

Every instance of Basic Law V

[V] {x: $\psi(x)\} = \{x : \chi(x)\} \leftrightarrow \forall x(\psi(x) \leftrightarrow \chi(x))$

is true in the model presented here. Recall that the formulæ $\psi(x)$ and $\chi(x)$ are subject to the restrictions concerning extension-terms, i.e. they contain neither free plural variables nor bound concept variables, whereas they may contain free concept variables and bound plural variables. It has to be shown that under every interpretation $\mathbf{I}$ of the free concept variables in $\psi(x)$ and $\chi(x)$, the equivalence holds.

From Section 4.2.1, recall that a formula $\mathbf{B}_i(x)$ containing free concept variables is equivalent to a formula $\mathbf{A}_i(x)$ which neither contains concept variables nor bound plural variables not already in $\mathbf{B}_i(x)$, under an interpretation $\mathbf{I}$ relative to x. Then, to the extension-term $\{x : \mathbf{B}_i(x)\}$ the denotation of the term $\{x : \mathbf{A}_i(x)\}$ may be assigned as its denotation, which was assigned through the $\omega \times \omega$ sequence in Section 4.1.1.

Suppose that, given an interpretation $\mathbf{I}$ relative to x of the free concept variables, the formula $\psi(x)$ is equivalent to a formula $\mathbf{A}_j(x)$ containing x as its only free variable; and suppose that, under $\mathbf{I}$ relative to x, the formula $\chi(x)$ is equivalent to a formula $\mathbf{A}_k(x)$ with x as its only free variable. Thus, assign to the terms $\{x : \psi(x)\}$ and $\{x : \chi(x)\}$ respectively the denotations of the terms $\{x : \mathbf{A}_j(x)\}$ and $\{x : \mathbf{A}_k(x)\}$ as their denotations. Both these latter have been assigned denotations in the $\omega \times \omega$ sequence. In the $\omega \times \omega$ sequence, either $\{x : \mathbf{A}_j(x)\}$ is prior to $\{x : \mathbf{A}_k(x)\}$ or the other way around. Suppose $\{x : \mathbf{A}_j(x)\}$ is prior to $\{x : \mathbf{A}_k(x)\}$, then $\{x : \mathbf{A}_j(x)\}$ and $\{x : \mathbf{A}_k(x)\}$ have the same denotation if, and only if, $\mathbf{A}_j(x)$ is equivalent to $\mathbf{A}_k(x)$ under $\mathbf{I}$ relative to x. Recall though that $\mathbf{A}_j(x)$ and $\mathbf{A}_k(x)$ are also equivalent to $\psi(x)$ and $\chi(x)$ respectively under $\mathbf{I}$ relative to x. So, if $\mathbf{A}_j(x)$

is equivalent to $\mathbf{A}_k(x)$, $\psi(x)$ is equivalent to $\chi(x)$ too, under $\mathbf{I}$ relative to x. Finally, the right-hand side of Basic Law V, $\forall x(\psi(x) \leftrightarrow \chi(x))$ is true if, and only if, $\psi(x)$ and $\chi(x)$ are equivalent under $\mathbf{I}$ relative to x.

4.4. Every Instance of PRC is True in This Model

The PRC axioms are all instances of

[PRC] $\exists F \forall x(Fx \leftrightarrow \psi(x))$,

where $\psi(x)$ contains neither F free, nor free plural variables, nor bound concept variables. It may contain, though, free concept variables and bound plural variables. It has to be shown that, for every interpretation $\mathbf{I}$ of its free concept variables, the class β defined by $\psi(x)$ under $\mathbf{I}$ belongs to $\pi(\mathbb{N})$, where β is the class defined by $\psi(x)$ relative to an interpretation $\mathbf{I}$ of the free concept variables in $\psi(x)$ if, and only if, for every n, $n \in \beta$ if, and only if, $\psi(\overline{n})$ is true under $\mathbf{I}$.

Given an interpretation $\mathbf{I}$ for the free variables in $\psi(x)$, there is a formula $\mathbf{A}_i(x)$ containing no free concept variables nor bound plural variables not already in $\psi(x)$, such that $\mathbf{A}_i(x)$ is equivalent to $\psi(x)$ under $\mathbf{I}$ relative to x. As $\mathbf{A}_i(x)$ is PRC-permissible, the class defined by $\mathbf{A}_i(x)$ is a class β_i such that $\beta_i \in \pi(\mathbb{N})$, on the grounds of the definition of $\pi(\mathbb{N})$. As $\psi(x)$ and $\mathbf{A}_i(x)$ are equivalent, they define identical classes under $\mathbf{I}$ relative to x. Thus, the class defined by $\psi(x)$ under $\mathbf{I}$ relative to x belongs to $\pi(\mathbb{N})$.

4.5. Every Instance of PLC is True in This Model

The PLC axioms are all the instances of

[PLC] $\exists X \forall y(y \eta X \leftrightarrow \phi(x))$,

where $\phi(x)$ does not to contain X free. $\phi(x)$ may contain both free and bound plural variables and both free and bound concept variables. It has to be shown that, for every interpretation $\mathbf{I}$ of the free concept and free plural variables in $\phi(x)$, the class defined by $\phi(x)$ under $\mathbf{I}$ belongs to $\wp(\mathbb{N})$.

First, I'll show that, for every interpretation $\mathbf{I}$ of its free plural variables, the class defined by $\phi(x)$ under $\mathbf{I}$ belongs to $\wp(\mathbb{N})$. Given an interpretation $\mathbf{I}$ for the free plural variables in $\phi(x)$, there is a formula $\mathbf{A}_i(x)$ containing no free plural variables nor bound plural variables not already in $\phi(x)$, such that $\mathbf{A}_i(x)$ is equivalent to $\phi(x)$ under $\mathbf{I}$ relative to x. Let the class defined by $\mathbf{A}_i(x)$ be a class $\alpha_i \in \wp(\mathbb{N})$. As $\phi(x)$ and $\mathbf{A}_i(x)$ are equivalent, they define the same class; and as the class defined by $\mathbf{A}_i(x)$ belongs to $\wp(\mathbb{N})$ by definition, the class defined by $\phi(x)$ under $\mathbf{I}$ belongs to $\wp(\mathbb{N})$.

Secondly, I'll consider the free and bound concept variables in $\phi(x)$ and fix an interpretation **I** for the free concept variables on the domain $\pi(\mathbb{N})$. Recall that, by definition, $\pi(\mathbb{N}) \subseteq \wp(\mathbb{N})$. Since the singular variables x in $\phi(x)$ range over $\mathbb{N}$, obviously the class defined by $\phi(x)$ belongs to $\wp(\mathbb{N})$.

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